

GENERIC SYZYGY SCHEMES

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ABSTRACT. For a finite dimensional vector space G we define the k -th generic syzygy scheme $\text{Gensyz}_k(G)$ by explicit equations. If $X \subset \mathbb{P}^n$ is cut out by quadrics and f is a p -th syzygy of rank $p + k + 1$ we show that the syzygy scheme $\text{Syz}(f)$ of f is a cone over a linear section of $\text{Gensyz}_k(G)$. We also give a geometric description of $\text{Gensyz}_k(G)$ for $k = 0, 1, 2$, in particular $\text{Gensyz}_2(G)$ is the union of a Plücker embedded Grassmannian and a linear space. From this we deduce that every smooth, non-degenerate projective curve $C \subset \mathbb{P}^n$ which is cut out by quadrics and has a p -th linear syzygy of rank $p + 3$ admits a rank 2 vector bundle \mathcal{E} with $\det \mathcal{E} = \mathcal{O}_C(1)$ and $h^0(\mathcal{E}) \geq p + 4$.

1. INTRODUCTION

Let $X \subset \mathbb{P}^n$ be a projective variety that is cut out by quadrics. One can then look at the linear strand of its minimal free resolution and ask whether a p -th linear syzygy f carries some geometric information about X . For this purpose Ehbauer [Ehb94] introduced the syzygy scheme $\text{Syz}(f)$, which is cut out by the quadrics involved in f . The syzygy scheme always contains X and can be explicitly calculated in some cases. Ehbauer studied this construction when X is a set of points in uniform position.

Another geometric invariant of a p -th syzygy f is the space G^* of linear forms involved in f . Its dimension is called the rank of f . Interesting syzygy varieties often arise from syzygies of low rank.

In [Sch91] Schreyer observed that for $p = 1$ the syzygy scheme $\text{Syz}(f)$ is always a cone over a linear section of a generic syzygy scheme Gensyz_k with $k = \text{rank } f - 2$ and gave explicit equations for Gensyz_k in this case. Eusen and Schreyer found a geometric description of these schemes for $k \in \{0, \dots, 4\}$ and $p = 1$ in [ES94].

In this paper we define more general generic syzygy schemes $\text{Gensyz}_k(G)$ by explicit equations depending on a finite dimensional vector space G . With these schemes we prove:

Theorem 3.4. Let $I \subset R$ be a homogeneous ideal generated by quadrics and f a p -th rank $p + k + 1$ linear syzygy of I . Then the syzygy scheme $\text{Syz}(f)$ is isomorphic to a cone over a linear section of $\text{Gensyz}_k(G)$ where G is the space of $(p - 1)$ -st syzygies involved in f .

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We also obtain a geometric description of $\text{Gensyz}_k(G)$ for $k = 0, 1, 2$ and arbitrary G . We show that $\text{Gensyz}_0(G)$ is always the union of a hypersurface with a point and that $\text{Gensyz}_1(G)$ is a Segre-embedded $\mathbb{P}^1 \times \mathbb{P}^{\text{rank } f - 1}$. The main new result of this paper is

Theorem 6.1. Let G be a g dimensional vector space, then

$$\text{Gensyz}_2(G) = \mathbb{G}(\mathbb{C} \oplus G^*, 2) \cup \mathbb{P}\left(\bigwedge^2 G^*\right) \subset \mathbb{P}(G^* \oplus \bigwedge^2 G^*),$$

where $\mathbb{G}(\mathbb{C} \oplus G^*, 2)$ is the Grassmannian of two dimensional quotient spaces of $\mathbb{C} \oplus G^*$. Moreover the second generic syzygy ideal I of G is reduced and saturated.

The geometric descriptions of $\text{Gensyz}_k(G)$ allow us to draw a number of conclusions:

Corollary 4.2. *Let $X \subset \mathbb{P}^n$ be a projective variety, I_X generated by quadrics and $f \in F_p$ a p -th syzygy of rank $p + 1$. Then X is either contained in a hyperplane or reducible.*

This result seems to be well known, but we include it since it follows directly from our methods.

Corollary 5.2. *Let $X \subset \mathbb{P}^n$ be a non-degenerate irreducible projective variety, I_X generated by quadrics and $f \in F_p$ a p -th syzygy of rank $p + 2$. Then the syzygy scheme $\text{Syz}(f)$ of f is a scroll of degree $p + 2$ and codimension $p + 1$.*

In particular a p -th syzygy of rank $p + 1$ implies the existence of a special pencil $|D|$ on X cut out by the fibers of the scroll. If X is a canonical curve $|D|$ has low Clifford index. These pencils are the ones that play a role in Green's conjecture [Gre84]. Our corollary above is therefore probably well known to experts in this field.

Our main new geometric result is

Theorem 6.7. *Let $C \subset \mathbb{P}^n$ be a smooth, irreducible non-degenerate curve. If C is cut out by quadrics and has a p -th syzygy f of rank $p + 3$, then there exists a rank 2 vector bundle \mathcal{E} on C with $\det \mathcal{E} = \mathcal{O}_C(1)$ and $h^0(\mathcal{E}) \geq p + 4$.*

In the case of a canonical curve these are rank 2 bundles with canonical determinant.

One can also use the methods of this paper to construct the Mukai-Lazarsfeld bundle on a $K3$ surface directly from a syzygy f . This is the vector bundle that played a central role in Voisin's proof of Green's conjecture [Voi02], [Voi03]. The Grassmannian used by Voisin in her proof is dual to the Grassmannian obtained as the generic syzygy scheme of f .

This paper is structured as follows. In Section 2 we recall what we need about syzygies, syzygy ideals and syzygy schemes. In Section 3 we define the generic syzygy varieties and show that every syzygy scheme is a cone over a linear section of a generic syzygy scheme. In the last three sections we describe the k -th generic syzygy varieties for $k = 0, 1, 2$ geometrically and study syzygies of rank $p + 1$, $p + 2$ and $p + 3$.

2. SYZYGIES, SYZYGIES IDEALS AND SYZYGIES SCHEMES

For the purpose of this paper let $R = \mathbb{C}[x_0, \dots, x_n]$ be the homogeneous coordinate ring of \mathbb{P}^n . With $R(-i)$ we denote R with its grading shifted, i.e. $R(-i)_j = R_{j-i}$. Often we abbreviate the space of linear polynomials $R_1 \subset R$ by V and write $\mathbb{P}^n = \mathbb{P}(V)$ using the Grothendieck notation.

Definition 2.1. Let $I \subset R$ be a homogeneous ideal, generated by quadrics, and

$$F_\bullet : I \leftarrow F_0 \otimes R(-2) \leftarrow \dots \leftarrow F_r \otimes R(-r-2)$$

the linear part of the minimal free resolution of I . The elements of F_i are called i -th *linear syzygies* of I .

Definition 2.2. Let $I \subset R$ be a homogeneous ideal, generated by quadrics and $f \in F_p$ a p -th linear syzygy. We define the *space of $(p-1)$ -st linear syzygies involved in f* as the smallest vector space $G \subset F_{p-1}$ such that there is a commutative diagram

$$\begin{array}{ccc} F_{p-1} \otimes R(-p-1) & \longleftarrow & F_p \otimes R(-p-2) \\ \uparrow & & \uparrow \\ G \otimes R(-p-1) & \longleftarrow & f \otimes R(-p-2). \end{array}$$

We define the *rank* of f as the dimension of G .

The above diagram extends to a map from the Koszul complex of G to the linear strand of I :

$$\begin{array}{ccccccc} I & \longleftarrow & F_0 \otimes R(-2) & \longleftarrow & \dots & \longleftarrow & F_p \otimes R(-p-2) \\ \uparrow & & \uparrow & & & & \uparrow \\ \bigwedge^{p+1} G \otimes R(-1) & \longleftarrow & \bigwedge^p G \otimes R(-2) & \longleftarrow & \dots & \longleftarrow & f \otimes R(-p-2). \end{array}$$

The image of $\bigwedge^p G$ in I is called the *syzygy ideal* I_f of f .

Remark 2.3. Observe that by dualizing and twisting the morphism

$$G \otimes R(-p-1) \leftarrow f \otimes R(-p-2)$$

from above, G^* is exhibited as a space of linear forms on \mathbb{P}^n . We therefore call G^* the *space of linear forms involved in f* .

Lemma 2.4. *In the map of complexes of Definition 2.2 all vertical maps are nonzero.*

Proof. Suppose there exists an integer k such that in the diagram

$$\begin{array}{ccc} F_{k-1} \otimes R(-k-1) & \longleftarrow & F_k \otimes R(-k-2) \\ \varphi_{k-1} \uparrow & & \uparrow \varphi_k \\ \bigwedge^{p-k+1} G \otimes R(-k-1) & \longleftarrow & \bigwedge^{p-k} G \otimes R(-k-2) \end{array}$$

the morphism φ_{k-1} is zero, but φ_k is nonzero. Then the image of φ_k is a free summand of $F_k \otimes R(-k-2)$ which maps to zero in the linear strand of the minimal free resolution of I . This contradicts the minimality of the resolution. \square

Corollary 2.5. *Let f be a p -th linear syzygy of $I \subset R$. Then $\text{rank } f \geq p+1$.*

Proof. If $\text{rank } f \leq p$ then $\bigwedge^{p+1} G$ vanishes and the first vertical map of the map of complexes in Definition 2.2 would have to be zero. \square

Definition 2.6. Let $I \subset R$ be an ideal generated by quadrics, $f \in F_p$ a p -th linear syzygy and I_f the syzygy ideal of f . Then the vanishing set $\text{Syz}(f) = V(I_f)$ is called the syzygy scheme associated to f .

Remark 2.7. Observe that $\text{Syz}(f) \subset \mathbb{P}^n$ is always a strict subset, since the syzygy ideal I_f is never empty by Lemma 2.4.

3. GENERIC SYZYGY SCHEMES

Definition 3.1. Let G be a vector space of dimension g and consider the ring $S = \mathbb{C}[G^* \oplus \bigwedge^k G^*]$. The ideal I defined by the natural inclusion

$$I = \bigwedge^{k+1} G^* \subset G^* \otimes \bigwedge^k G^* \subset S^2\left(G^* \oplus \bigwedge^k G^*\right) \subset S$$

is called the k -th generic syzygy ideal of G . Its vanishing set $\text{Gensyz}_k(G)$ is called the k -th generic syzygy scheme of G .

Proposition 3.2. *Let I be the k -th generic syzygy ideal of G . Then the linear strand of I has the last $g-k$ steps of the Koszul complex associated to G^* as a natural subcomplex, i.e we have a commutative diagram:*

$$\begin{array}{ccccccc} I & \longleftarrow & F_0 \otimes S(-2) & \longleftarrow & \cdots & \longleftarrow & F_{g-k-1} \otimes S(-g+k-1) \\ & & \uparrow & & & & \uparrow \\ & & \bigwedge^{k+1} G^* \otimes S(-2) & \longleftarrow & \cdots & \longleftarrow & \bigwedge^g G^* \otimes S(-g+k-1) \end{array}$$

Proof. The inclusion $I = \bigwedge^{k+1} G^* \subset G^* \otimes \bigwedge^k G^* \subset S^2\left(G^* \oplus \bigwedge^k G^*\right) \subset S$ induces a commutative diagram of free S -modules

$$\begin{array}{ccc} I & \longleftarrow & F_0 \otimes S(-2) \\ \uparrow & & \parallel \\ \bigwedge^k G^* \otimes S(-1) & \longleftarrow & \bigwedge^{k+1} G^* \otimes S(-2). \end{array}$$

The top arrow is resolved by the minimal free resolution of I and the bottom arrow by the rest of the Koszul complex. Since both complexes are exact and minimal, the maps above lift to a map of complexes. This map is injective in each new step since it is injective in the F_0 step. For degree reasons, the image of this map of complexes must lie in the linear strand of I . \square

Corollary 3.3. *The k -th generic syzygy scheme of G has a natural 1-dimensional space of rank g linear syzygies in step $g - k - 1$. The space of $(g - k - 2)$ -nd syzygies involved in anyone of these is isomorphic to G .*

Proof. The $(g - k - 1)$ -st syzygies given by Proposition 3.2 have rank at most g since $\bigwedge^{g-1} G^* \cong G$ has dimension g . The rank of these syzygies cannot be smaller, since the last map of the Koszul complex is surjective in degree $g - k$. \square

Theorem 3.4. *Let $I \subset R$ be a homogeneous ideal generated by quadrics and f a p -th rank $p + k + 1$ linear syzygy of I . Then the syzygy scheme $\text{Syz}(f)$ is isomorphic to a cone over a linear section of $\text{Gensyz}_k(G)$ where G is the space of $(p - 1)$ -st syzygies involved in f .*

Proof. We have the map of complexes

$$\begin{array}{ccccccc} R & \longleftarrow & F_0 \otimes R(-2) & \longleftarrow & \cdots & \longleftarrow & F_p \otimes R(-p-2) \\ \uparrow \alpha & & \uparrow & & & & \uparrow \\ \bigwedge^{p+1} G \otimes R(-1) & \longleftarrow & \bigwedge^p G \otimes R(-2) & \longleftarrow & \cdots & \longleftarrow & f \otimes R(-p-2) \end{array}$$

from Definition 2.2. Consider the map

$$\varphi: G^* \oplus \bigwedge^k G^* \rightarrow V$$

given by mapping the elements of G^* to their corresponding linear forms and the elements of $\bigwedge^k G^* = \bigwedge^{p+1} G$ to their images under the map α . The induced diagram

$$\begin{array}{ccccccc} R & \longleftarrow & F_0 \otimes R(-2) & \longleftarrow & & & \\ \uparrow \alpha & & \uparrow & & \searrow & & \\ \bigwedge^{p+1} G \otimes R(-1) & \longleftarrow & \bigwedge^p G \otimes R(-2) & \longleftarrow & S & \longleftarrow & \bigwedge^{k+1} G^* \otimes S(-2) \\ & & & & \uparrow & & \parallel \\ & & & & \bigwedge^k G^* \otimes S(-1) & \longleftarrow & \bigwedge^{k+1} G^* \otimes S(-2) \end{array}$$

and its degree 2 part

$$\begin{array}{ccccccc} S^2 V & \longleftarrow & F_0 & \longleftarrow & & & \\ \uparrow \alpha & & \uparrow & & \searrow & & \\ \bigwedge^{p+1} G \otimes V & \longleftarrow & \bigwedge^p G & \longleftarrow & S^2(G^* \oplus \bigwedge^k G^*) & \longleftarrow & \bigwedge^{k+1} G^* \\ & & & & \uparrow & & \parallel \\ & & & & \bigwedge^k G^* \otimes (G^* \oplus \bigwedge^k G^*) & \longleftarrow & \bigwedge^{k+1} G^* \end{array}$$

shows that φ maps the k -th generic syzygy ideal surjectively to the syzygy ideal I_f of f . Projectively the image of φ defines a linear subspace

$$\mathbb{P}(\operatorname{Im} \varphi) \subset \mathbb{P}(G^* \oplus \bigwedge^k G^*).$$

The calculation above shows that $\operatorname{Syz}(f)$ is a cone over $\mathbb{P}(\operatorname{Im} \varphi) \cap \operatorname{Gensyz}_k(G)$ with vertex $V(\operatorname{Im} \varphi) \subset \mathbb{P}(V)$. \square

4. REDUCIBLE SYZYGIES

Proposition 4.1. *Let G be a g dimensional vector space, then*

$$\operatorname{Gensyz}_0(G) \cong \mathbb{P}(G^*) \cup \mathbb{P}(\mathbb{C}) \subset \mathbb{P}(G^* \oplus \mathbb{C}),$$

i.e. $\operatorname{Gensyz}_0(G)$ is the union of a hyperplane and a point. Moreover the generic syzygy ideal of I of $\operatorname{Gensyz}_0(G)$ is reduced and saturated.

Proof. The ideal of the hyperplane $\mathbb{P}(G^*) \cong \mathbb{P}^{g-1}$ is generated by the linear forms in $\bigwedge^0 G^* \cong \mathbb{C}$. The ideal of the point $\mathbb{P}(\bigwedge^0 G^*) \cong \mathbb{P}^0$ is generated by the linear forms in G^* . Since the two ideals involve different sets of variables, their intersection is the same as their product:

$$I_{\mathbb{P}^{g-1}} \cap I_{\mathbb{P}^0} = (G^*) \cap \left(\bigwedge^0 G^* \right) = (G^*) \cdot \left(\bigwedge^0 G^* \right) = (G^* \otimes \bigwedge^0 G^*) = \left(\bigwedge^1 G^* \right)$$

This is the 0-th generic syzygy ideal of G . \square

Corollary 4.2. *Let $X \subset \mathbb{P}^n$ be a projective variety, I_X generated by quadrics and $f \in F_p$ a p -th syzygy of rank $p+1$. Then X is either contained in a hyperplane or reducible.*

Proof. By Theorem 3.4 and Proposition 4.1 $\operatorname{Syz}(f)$ is a cone over a linear section of a hyperplane and a point. Since $\operatorname{Syz}(f)$ can not contain all of \mathbb{P}^n by Remark 2.7, $\operatorname{Syz}(f) \subset \mathbb{P}^n$ must be the union of a hyperplane and possibly a second linear subspace. Since X is contained in $\operatorname{Syz}(f)$ it must be either reducible or contained in one of the two linear subspaces. \square

Definition 4.3. Let $X \subset \mathbb{P}^n$ be a projective scheme, whose ideal is cut out by quadrics. A p -th linear syzygy of X is called *reducible*, if it has rank $p+1$.

5. SCROLLAR SYZYGIES

Theorem 5.1. *Let G be a g dimensional vector space, then*

$$\operatorname{Gensyz}_1(G) = \mathbb{P}(G^*) \times \mathbb{P}^1 \subset \mathbb{P}(G^* \oplus G^*).$$

Moreover the second generic syzygy ideal I of G is reduced and saturated.

Proof. Observe that $G^* \otimes (\mathbb{C} \oplus \mathbb{C}) = G^* \oplus G^*$. We can therefore consider the Segre embedding

$$\mathbb{P}^{g-1} \times \mathbb{P}^1 = \mathbb{P}(G^*) \times \mathbb{P}(\mathbb{C} \oplus \mathbb{C}) \subset \mathbb{P}(G^* \oplus G^*).$$

The ideal of $\mathbb{P}^{g-1} \times \mathbb{P}^1$ is generated by the Segre quadrics:

$$I_{\mathbb{P}^{g-1} \times \mathbb{P}^1} = \left(\bigwedge^2 G^* \otimes \bigwedge^2 (\mathbb{C} \oplus \mathbb{C}) \right) = \left(\bigwedge^2 G^* \right)$$

This is the first generic syzygy ideal of G . \square

Corollary 5.2. *Let $X \subset \mathbb{P}^n$ be a non degenerate irreducible projective variety, I_X generated by quadrics and $f \in F_p$ a p -th syzygy of rank $p+2$. Then the syzygy scheme $\text{Syz}(f)$ of f is a scroll of degree $p+2$ and codimension $p+1$.*

Proof. Let G be the $g = p+2$ dimensional space of $(p-1)$ -st syzygies involved in f . By theorem 3.4 the syzygy scheme $\text{Syz}(f)$ is a linear section of a cone over $\mathbb{P}^{p+1} \times \mathbb{P}^1$. Since $\mathbb{P}^{p+1} \times \mathbb{P}^1$ has codimension $p+1$ and degree $p+2$ in $\mathbb{P}(G^* \oplus \Lambda^1 G^*)$ we only have to prove that this intersection is of expected codimension. By Eisenbud [Eis95, Ex. A2.19] this is the case if the matrix M whose 2×2 -minors cut out $\mathbb{P}^{p+1} \times \mathbb{P}^1$ remains 1-generic after we apply the map

$$\varphi: G^* \oplus \Lambda^1 G^* \rightarrow V$$

from the proof of Theorem 3.4.

If $\varphi(M)$ is not 1-generic, we can choose bases of G^* and $\mathbb{C} \oplus \mathbb{C}$ such that $\varphi(M)$ has the form

$$M = \begin{pmatrix} l_1 & \cdots & l_i & l_{i+1} & \cdots & l_g \\ a_1 & \cdots & a_i & 0 & \cdots & 0 \end{pmatrix}$$

with l_1, \dots, l_{p+1} a basis of G^* and a_1, \dots, a_i linearly independent. Since the syzygy ideal I_f cannot be empty by Lemma 2.4, i has to be at least 1. In this situation I_f contains the 2×2 minor

$$\det \begin{pmatrix} l_1 & l_g \\ a_1 & 0 \end{pmatrix} = l_g \cdot a_1$$

which implies that X must be reducible or degenerate. This contradicts our assumptions. \square

Definition 5.3. Let $X \subset \mathbb{P}^n$ be a projective scheme, whose ideal is cut out by quadrics. A p -th linear syzygy of X is called *scrollar*, if it has rank $p+2$.

Example 5.4. Let $C \subset \mathbb{P}^{g-1}$ be a non hyperelliptic canonical curve of genus g and $|D|$ a pencil of Clifford index $\text{cliff}(D) = g - p - 3$. The p -th syzygy of C constructed by the method of Green and Lazarsfeld in [GL84] is scrollar.

With the above geometric description of scrollar syzygy varieties one can prove the following well known converse of the Green-Lazarsfeld construction:

Proposition 5.5. *Let $C \subset \mathbb{P}^{g-1}$ be a non hyperelliptic canonical curve of genus g and $f \in F_p$ a p -th scrollar syzygy. Then there exists a linear system $|D|$ on C with Clifford index $\text{cliff}(D) \leq g - p - 3$.*

Proof. Let G^* be the $p + 2$ dimensional space of linear forms involved in f . Then the syzygy scheme $\text{Syz}(f)$ of f is a scroll that contains C and has the vanishing set $V(G^*)$ as a fiber. Set $D = C \cap V(G^*)$. Since $C \subset \mathbb{P}^{g-1}$ is non degenerate, D is a divisor on C . We consider the linear system $|D|$. Since D is cut out by the ruling of $\text{Syz}(f)$ we have $h^0(D) \geq 2$. Also $h^0(K - D) \geq p + 2$ since the linear forms in G^* cut out canonical divisors of C that contain D . Riemann-Roch now gives:

$$\begin{aligned} \text{cliff } D &:= d - 2r = (h^0(D) - h^0(K - D) - 1 + g) - 2h^0(D) + 2 = \\ &= g + 1 - h^0(D) - h^0(K - D) \geq g + 1 - 2 - (p + 2) = g - p - 3. \end{aligned}$$

□

Remark 5.6. For general k -gonal canonical curves C Green's conjecture is equivalent to the claim that every step of the linear strand C contains at least one scrollar syzygy. This was recently shown by Voisin [Voi02], [Voi03].

More generally one can make the following conjecture

Conjecture 5.7 (Generic Geometric Syzygy Conjecture). *Let $C \subset \mathbb{P}^{g-1}$ be a general canonical curve of genus g . Then for every p the space of p -th linear syzygies of C is spanned by scrollar syzygies.*

This conjecture is known for $p = 1$ when $g \neq 8$ and for $p = 2$ when $g = 8$ by [vB00] and [vB02].

6. GRASSMANNIAN SYZYGIES

Theorem 6.1. *Let G be a g dimensional vector space, then*

$$\text{Gensyz}_2(G) = \mathbb{G}(\mathbb{C} \oplus G^*, 2) \cup \mathbb{P}\left(\bigwedge^2 G^*\right) \subset \mathbb{P}\left(G^* \oplus \bigwedge^2 G^*\right),$$

where $\mathbb{G}(\mathbb{C} \oplus G^*, 2)$ is the Grassmannian of two dimensional quotient spaces of $\mathbb{C} \oplus G^*$. Moreover the second generic syzygy ideal I of G is reduced and saturated.

Proof. Observe that $\bigwedge^2(\mathbb{C} \oplus G^*) = G^* \oplus \bigwedge^2 G^*$. We can therefore consider the Plücker embedding

$$\mathbb{G} := \mathbb{G}(\mathbb{C} \oplus G^*, 2) \subset \mathbb{P}\left(G^* \oplus \bigwedge^2 G^*\right)$$

and the ideal of the Grassmannian \mathbb{G} which is generated by 4×4 -pfaffians of a skew symmetric matrix. More precisely:

$$I_{\mathbb{G}} = \left(\bigwedge^4(\mathbb{C} \oplus G^*)\right) = \left(\bigwedge^3 G^* \oplus \bigwedge^4 G^*\right) \subset S^2\left(G^* \oplus \bigwedge^2 G^*\right).$$

On the other hand $\mathbb{P}(\bigwedge^2 G^*) \cong \mathbb{P}^{\binom{g}{2}-1} =: \mathbb{P}$ is cut out by the linear forms in G^* , so $I_{\mathbb{P}} = (G^*)$. To prove the theorem we calculate the intersection of

these two irreducible ideals:

$$\begin{aligned} I_{\mathbb{P}} \cap I_{\mathbb{G}} &= (G^*) \cap \left(\bigwedge^3 G^* \oplus \bigwedge^4 G^* \right) \\ &= ((G^*) \cap \bigwedge^3 G^*) + ((G^*) \cap \bigwedge^4 G^*). \end{aligned}$$

Now the quadrics in the ideal (G^*) are given by the image of

$$G^* \otimes (G^* \oplus \bigwedge^2 G^*) \rightarrow S^2(G^* \oplus \bigwedge^2 G^*),$$

i.e

$$(I_{\mathbb{P}})_2 = S^2 G^* \oplus G^* \otimes \bigwedge^2 G^* = S^2 G^* \oplus \bigwedge^3 G^* \oplus \bigwedge^{2,1} G^*.$$

This shows that $(\bigwedge^3 G^*)$ is contained in (G^*) . For the second intersection of ideals notice that $\bigwedge^4 G^*$ is contained in $S^2(\bigwedge^2 G^*)$. So the generators of $(\bigwedge^4 G^*)$ and (G^*) involve different sets of variables and the intersection of the two ideals is the same as their product:

$$\begin{aligned} (G^*) \cap \left(\bigwedge^4 G^* \right) &= (G^*) \cdot \left(\bigwedge^4 G^* \right) = (G^* \otimes \bigwedge^4 G^*) = \\ &= \left(\bigwedge^5 G^* \oplus \bigwedge^{4,1} G^* \right) \subset G^* \otimes S^2(\bigwedge^2 G^*) \subset S^3(G^* \oplus \bigwedge^2 G^*). \end{aligned}$$

On the other hand the cubics of $(\bigwedge^3 G^*)$ contain

$$\bigwedge^3 G^* \otimes \bigwedge^2 G^* = \bigwedge^5 G^* \oplus \bigwedge^{4,1} G^* \subset G^* \otimes S^2(\bigwedge^2 G^*).$$

Since these representations occur only once in $G^* \otimes S^2(\bigwedge^2 G^*)$ they must be the ones that generate the product of ideals above. In total we have shown

$$I_{\mathbb{P}} \cap I_{\mathbb{G}} = \left(\bigwedge^3 G^* \right)$$

which is the second generic syzygy ideal of G . \square

Definition 6.2. Let $X \subset \mathbb{P}^n$ be a projective scheme, whose ideal is cut out by quadrics. A p -th linear syzygy of X is called *grassmannian*, if it has rank $p + 3$.

Example 6.3. Let X be a $K3$ -surface of sectional genus g in \mathbb{P}^g with Picard group generated by a general hyperplane section H . Then X has grassmannian p -th syzygies for $p \leq \frac{g-4}{2}$.

Proof. X is cut out by quadrics. Since X is irreducible and non-degenerate, X has no reducible syzygies and does not lie on quadrics of rank 2 or 1. X can also not lie on a quadrics of rank 4 or 3, since in this case the rulings of the quadrics would cut out divisors of degree smaller than H on X . Hence, because scrolls are cut out by 2×2 minors of rank at most 4, X can have no scrollar syzygies.

Now intersect X with a general hyperplane H . Then $X \cap H = C \subset \mathbb{P}^{g-1}$ is a canonical curve whose minimal free resolution is the restriction of the minimal free resolution of X to H . By the construction of Green and Lazarsfeld

C has scrollar p -th syzygies for $p \leq \frac{g-4}{2}$. The rank of a syzygy f can fall by at most one when restricting to a general hyperplane (i.e. when the linear form defining H is involved in f). Since X has no scrollar syzygies, the scrollar syzygies of C must come from grassmannian syzygies of X . \square

We now describe some geometric consequences of grassmannian syzygies. For this let \mathcal{Q} be the universal rank 2 quotient bundle on the Grassmannian $\mathbb{G} = \mathbb{G}(\mathbb{C} \oplus G^*, 2)$. The global sections of \mathcal{Q} are given by $H^0(\mathbb{G}, \mathcal{Q}) = \mathbb{C} \oplus G^*$.

Lemma 6.4. *Let $s \in H^0(\mathbb{G}, \mathcal{Q})$ be a global section and I_s the ideal of its vanishing locus on \mathbb{G} . Then I_s is generated by hyperplane sections of \mathbb{G} , more precisely*

$$I_s = (s \wedge H^0(\mathbb{G}, \mathcal{Q})).$$

Proof. Consider the Koszul complex associated to s :

$$0 \rightarrow \mathcal{O}_{\mathbb{G}} \xrightarrow{s} \mathcal{Q} \rightarrow I_s \otimes \bigwedge^2 \mathcal{Q} \rightarrow 0$$

Taking cohomology shows $(s \wedge H^0(\mathbb{G}, \mathcal{Q})) \subset I_s$. Since \mathcal{Q} is globally generated, the converse also follows. \square

Remark 6.5. Observe that for a section $s \in \mathbb{C} \subset \mathbb{C} \oplus G^* = H^0(\mathbb{G}, \mathcal{Q})$ we have $I_{\mathbb{G}} + I_{\mathbb{P}} = I_s$. In other words a grassmannian syzygy f defines up to a constant a section of \mathcal{Q} .

Lemma 6.6. *Let $X \subset \mathbb{P}(V)$ be a projective variety cut out by quadrics, f a p -th grassmannian syzygy of X , G the space of $(p-1)$ -st syzygies involved in f , and $\varphi: G^* \oplus \bigwedge^2 G^* \rightarrow V$ the induced map. Then the natural map*

$$H^0(\mathbb{G}, \mathcal{Q}) \rightarrow H^0(\mathbb{G} \cap \mathbb{P}(\text{Im } \varphi), \mathcal{Q}|_{\mathbb{G} \cap \mathbb{P}(\text{Im } \varphi)})$$

is injective.

Proof. By construction $\text{Im } \varphi$ contains G^* so the non-zero elements of G^* are not contained in $I_{\mathbb{P}(\text{Im } \varphi)}$. On the other hand the vanishing ideal

$$I_s = (s \wedge (\mathbb{C} \oplus G^*))$$

contains the whole space $\mathbb{C} \wedge G^* = G^*$ if $s \in \mathbb{C}$, or a non-zero element of $G^* \wedge \mathbb{C} = G^*$ if $s \in G^*$. So I_s can never be contained in $I_{\mathbb{P}(\text{Im } \varphi)}$ and $H^0(\mathcal{Q} \otimes I_{\mathbb{G} \cap \mathbb{P}(\text{Im } \varphi)/\mathbb{G}}) = 0$. The proposition then follows from the exact sequence

$$0 \rightarrow \mathcal{Q} \otimes I_{\mathbb{G} \cap \mathbb{P}(\text{Im } \varphi)/\mathbb{G}} \rightarrow \mathcal{Q} \rightarrow \mathcal{Q}|_{\mathbb{G} \cap \mathbb{P}(\text{Im } \varphi)/\mathbb{G}} \rightarrow 0.$$

\square

Theorem 6.7. *Let $C \subset \mathbb{P}^n$ be a smooth, irreducible non-degenerate curve. If C is cut out by quadrics and has a p -th grassmannian syzygy f , then there exists a rank 2 vector bundle \mathcal{E} on C with $\det \mathcal{E} = \mathcal{O}_C(1)$ and $h^0(\mathcal{E}) \geq p+4$.*

Proof. Let $\text{Syz}(f)$ be the syzygy scheme of f . By Theorems 3.4 and 6.1 $\text{Syz}(f)$ is a cone over a linear section of $\mathbb{G}(p+4, 2) \cup \mathbb{P}^{\binom{p+3}{2}-1}$. Now $\text{Syz}(f)$ contains C and C is irreducible and non-degenerate, so C must be contained in a cone Y over a linear section of \mathbb{G} . The universal quotient bundle \mathcal{Q} on

\mathbb{G} restricts to $\mathbb{G} \cap \mathbb{P}(\mathrm{Im} \phi)$ and pulls back to a rank 2 vector bundle \mathcal{Q}_{Y° on $Y^\circ = Y \setminus V(\mathrm{Im} \phi)$. If C does not intersect the vertex $V(\mathrm{Im} \phi)$ of Y the restriction of \mathcal{Q}_{Y° to C is a vector bundle \mathcal{E} .

If C intersects the vertex of Y in a divisor, we consider the blowup \tilde{Y} of Y in the vertex. \mathcal{Q} then pulls back to a rank 2 vector bundle $\mathcal{Q}_{\tilde{Y}}$ on \tilde{Y} . Since C is smooth the strict transform \tilde{C} of C is isomorphic to C and $\mathcal{Q}_{\tilde{Y}}$ restricts to a rank 2 vector bundle \mathcal{E} on $\tilde{C} \cong C$.

Finally C can not be contained in the vertex of Y since C is non-degenerate.

By Lemma 6.6 we have $h^0(\mathcal{Q}|_{\mathbb{G} \cap \mathbb{P}(\mathrm{Im} \phi)}) \geq p + 4$. These sections extend to Y° . By Lemma 6.4 the zero loci of sections of \mathcal{Q} are cut out by linear forms and their closures contain the vertex of Y . Since X is non-degenerate it can not lie in one of these zero loci, so all sections of \mathcal{Q} descend to sections of \mathcal{E} . \square

Example 6.8. Our method can in some cases also be used to obtain vector bundles on varieties of higher dimension. Let for example $X \subset \mathbb{P}^g$ be a K3 surface of even sectional genus $g = 2k$ whose Picard group is generated by a general hyperplane section. Then X has a grassmannian $(k - 2)$ -nd syzygy by the argument of Example 6.3. One can show that in this case the map

$$\varphi: G^* \oplus \bigwedge^2 G^* \rightarrow V$$

is surjective. Therefore $\mathrm{Syz}(f)$ is not a cone, and \mathcal{Q} restricts to a rank 2 vector bundle \mathcal{E} on X with $\det \mathcal{E} = \mathcal{O}_X(1)$ and $h^0(\mathcal{E}) \geq k + 2$. This is the Mukai-Lazarsfeld bundle used by Voisin in her proof of Green's conjecture [Voi02].

This example leads us to ask

Question 6.9. *Let $X \subset \mathbb{P}^n$ be a surface cut out by quadrics whose Picard group is generated by a general hyperplane section. Does every step of the linear strand of X contain a grassmannian syzygy?*

Remark 6.10. Voisin's Theorem about the syzygies of K3 surfaces in [Voi02] prove that the answer to this question is "yes" in the case of K3 surfaces $X \subset \mathbb{P}^g$ with sectional genus $g = 2k$.

Even more generally we ask

Question 6.11. *Let $X \subset \mathbb{P}^n$ be a surface cut out by quadrics whose Picard group is generated by a general hyperplane section. Is the space of p -th linear syzygies of X spanned by grassmannian syzygies?*

Remark 6.12. The answer to this question is "yes" for general K3 surfaces $X \subset \mathbb{P}^g$ with sectional genus $g \leq 8$ by the methods of [vB02]

REFERENCES

- [Ehb94] S. Ehbauer. Syzygies of points in projective space and applications. In F. Orecchia, editor, *Zero-dimensional schemes. Proceedings of the international conference held in Ravello, Italy, June 8-13, 1992*, pages 145–170, Berlin, 1994. de Gruyter.

- [Eis95] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*. Graduate Texts in Mathematics 150. Springer, 1995.
- [ES94] F. Eusei and F.O. Schreyer. A remark to a conjecture of Paranjape and Ramanan. <http://www.math.uni-sb.de/~ag-schreyer/DE/publikationen.html>, 1994.
- [GL84] M. Green and R. Lazarsfeld. The non-vanishing of certain Koszul cohomology groups. *J. Diff. Geom.*, 19:168–170, 1984.
- [Gre84] M.L. Green. Koszul cohomology and the geometry of projective varieties. *J. Differential Geometry*, 19:125–171, 1984.
- [Sch91] F.O. Schreyer. A standard basis approach to syzygies of canonical curves. *J. reine angew. Math.*, 421:83–123, 1991.
- [vB00] H.-Chr. Graf v. Bothmer. *Geometrische Syzygien von kanonischen Kurven*. Dissertation, Universität Bayreuth, 2000.
- [vB02] H.-Chr. Graf v. Bothmer. Geometric syzygies of Mukai varieties and general canonical curves with genus ≤ 8 . math.AG/0202133, 2002.
- [Voi02] C. Voisin. Green’s generic syzygy conjecture for curves of even genus lying on a $K3$ surface. *J. Eur. Math. Soc. (JEMS)*, 4(4):363–404, 2002.
- [Voi03] C. Voisin. Green’s canonical syzygy conjecture for generic curves of odd genus. math.AG/0301359, 2003.

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